Proceedings of Machine Learning Research vol xxx:1–18, 2021

Tight sampling and discarding bounds for scenario programs with an arbitrary number of removed samples

Licio Romao LICIO.ROMAO@ENG.OX.AC.UK
Kostas Margellos KOSTAS.MARGELLOS@ENG.OX.AC.UK
Antonis Papachristodoulou ANTONIS@ENG.OX.AC.UK
Department of Engineering Science, University of Oxford, Parks Road, OX1 3PJ, Oxford, United Kingdom

Abstract

The so-called scenario approach offers an efficient framework to address uncertain optimisation problems with uncertainty represented by means of scenarios. The sampling-and-discarding approach within the scenario approach literature allows the decision maker to trade feasibility to performance. We focus on a removal scheme composed by a cascade of scenario programs that removes at each stage a superset of the support set associated to the optimal solution of each of these programs. This particular removal scheme yields a scenario solution with tight guarantees on the probability of constraint violation; however, existing analysis restricts the number of discarded scenarios to be a multiple of the dimension of the optimisation problem. Motivated by this fact, this paper presents pathways to extend the theoretical analysis of this removal scheme. We first provide an extension for a restricted class of scenarios programs for which tight bounds can be obtained, and then we provide a conservative bound on the probability of constraint violation that is valid for any scenario program and an arbitrary number of removed scenarios, which is, however, not tight.

1. Introduction

Data abound in modern applications, and this can be leveraged to boost robustness against uncertainty. In the past decades, new research directions have sprung from this fact, and are now shaping the theoretical foundation of several disciplines, including control theory and machine learning. Under this scenario, several data-driven algorithms came to prominence. In this paper we focus on a specific randomised technique, called the scenario approach theory (Calafiore and Campi (2005, 2006); Campi and Garatti (2008, 2011)), to study uncertain optimisation problems that provides guarantees on the probability of constraint violation associated to the optimal solution of an optimisation problem whose constraints are enforced on scenarios corresponding to available data.

The standard scenario approach theory is concerned with guarantees on the probability of constraint violation for the optimal solution of a scenario program. One of its main results Campi and Garatti (2008) consists of a distribution-free upper bound for such violation that involves the number of samples, dimension of the underlying optimisation problem, and desired level of violation. This upper bound can then be exploited by the decision maker to decide the number of samples required to produce a solution with prescribed feasibility properties. The scenario approach theory then enables us to assess the risk of our decision by means of data, thus consisting of a data-driven approach to uncertain optimisation. Another relevant interpretation of the results of Campi and Garatti (2008) is their connection with the well-known chance-constrained formulation (Prékopa (2003); Nemirovski and Shapiro (2006); Pagnoncelli et al. (2009)) of optimisation problems. The scenario approach theory can then be interpreted as generating, with high probability, a feasible solution to chance-constrained problems which is, in general, not optimal. Several papers have studied the optimality gap between the chance-constrained formulation and its scenario approximation (Calafiore and Campi (2006); Campi and Garatti (2011); Esfahani et al. (2015)).

Notwithstanding the advances due to the results in Campi and Garatti (2008), and as price for their generality, the performance of a scenario program in terms of cost may be conservative. To this end, papers Campi and Garatti (2011) and Calafiore (2010) characterised the probability of violation of a scenario program in which the decision maker is allowed to discard some of the original scenarios. As the feasible set of a scenario program is enlarged when constraints are discarded, the results in Campi and Garatti (2011) and Calafiore (2010), known as the sampling-and-discarding approach or scenario theory with constraint removal, enable the decision maker to reduce the conservatism of a scenario solution, while keeping the probability of constraint violation under control. An interesting feature of the bound proposed in Campi and Garatti (2011) is the fact that it holds true for any removal scheme and, similarly to the bound in Campi and Garatti (2008), is distribution-free. However, in contrast with Campi and Garatti (2008), which holds with equality for a class of scenario programs called fully-supported, the bound in Campi and Garatti (2011) is not tight.

Recently, the authors in Romao et al. (2020a) and Romao et al. (2020b) have studied a specific removal scheme and provided a bound on the probability of constraint violation that outperforms the one in Campi and Garatti (2011). These papers also show tightness of the proposed bound by providing a class of scenario programs that achieves this bound with equality. The removal scheme analysed in Romao et al. (2020a) is composed by a cascade of scenario programs where at each stage a superset of the support scenarios associated to the optimal solution is removed. However, their analysis restricts the number of discarded scenarios to be a multiple of the dimension of the optimisation problem.

The main contribution of this paper is to explore the extent to which the analysis of this particular removal scheme can be extended to allow for arbitrary discarded scenarios. First, we characterise the class of scenario programs that permits such arbitrary removal. We show that this coincides with the class of problems that led to tight bounds in Romao et al. (2020a). For general scenario programs, we argue that no tight results can be obtained without exploring additional structure of the problem and provide an upper bound on the probability of constraint violation that improves upon the bound in Campi and Garatti (2011). Moreover, if we are dealing with a min-max scenario program (that still fits in our class), we also present another alternative to remove scenarios that combines our removal procedure with the strategy presented in Care et al. (2015) and Garatti et al. (2019).

This paper is organised as follows. The main concepts of the scenario approach theory and the removal strategy analysed in Romao et al. (2020a) are presented in Section 2. The main results of this paper, which include a generalisation of the removal procedure of Romao et al. (2020a) to an arbitrary discarded scenarios for a subclass of scenario programs, are presented in Section 3. Section 4 concludes the paper and provides directions for future work.

2. Problem statement

Let Δ represent the uncertainty space, which is endowed with a σ-algebra $\mathcal{F}$, and suppose that there is an unknown probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$ defined over $\mathcal{F}$. The triple $(\Delta, \mathcal{F}, \mathbb{P})$ is called a probability measure[1] space. Fix $\epsilon \in (0, 1)$, and suppose that we want to find the optimal solution of the chance-constrained problem

$$\begin{aligned} \underset{x\in\mathcal{X}}{\text{minimise}} \quad & c^\top x \\ \text{subject to} \quad & \mathbb{P}\{\delta : g(x,\delta) > 0\} \leq \epsilon, \end{aligned} \tag{1}$$

1. We refer the reader to (Salamon, 2016, Chapter 1) for an introduction to these measure theoretic concepts.

where $\mathcal{X} \subset \mathbb{R}^d$ is a closed, convex set with non-empty interior and function $g : \mathbb{R}^d \times \Delta \to \mathbb{R}$ is measurable[2] in the second argument for each $x \in \mathcal{X}$. The feasible set of (1) comprises of points $x \in \mathbb{R}^d$ with the property that the probability of a δ sampled from $\mathbb{P}$ violates the constraint $g(x, \delta) \leq 0$ is smaller than ϵ. Unfortunately, the feasible set of (1) may be non-convex, even when the function $g(\cdot, \delta) : \mathbb{R}^d \to \mathbb{R}$ is convex for all $\delta \in \Delta$, so problem (1) is in general hard to solve. One common approach to obtain a solution to problem (1) is by means of a randomised approximation called the scenario approach with discarded constraints, which studies the optimal solution of the scenario program

$$\begin{aligned} &\underset{x \in \mathcal{X}}{\text{minimise}} \quad c^\top x \\ &\text{subject to} \quad g(x, \delta_i) \leq 0, \text{ with } \delta_i \in S \setminus R(S), \end{aligned} \tag{2}$$

where $\mathcal{X}$ and g are defined as above, with the additional assumption that $g(\cdot, \delta) : \mathbb{R}^d \to \mathbb{R}$ is convex for all $\delta \in \Delta$, $S = \{\delta_1, \dots, \delta_m\}$ is a set of i.i.d. samples from $\mathbb{P}$ (also referred as scenarios), and $R(S)$ contains scenarios that have been discarded by a removal procedure. The dependence of the removed scenarios on S is made explicit.

Some concepts at the core of the scenario approach theory which will be important to our developments are presented in the sequel.

Definition 1 (Support constraints) *Consider the scenario optimization problem* (2)*. A scenario in* $S \setminus R(S)$ *is said to be a support constraint (or support scenario) if its removal changes the optimal solution of* (2)*. The set of all support constraints is called the support set of* (2)*, and will be denoted by* $\text{supp}(x^\star(S))$ *throughout this paper.*

Definition 2 (Fully-supported problems) *A scenario optimization problem* (2) *is said to be fully-supported if for all* $m \in \mathbb{N}$ *the cardinality of the support set is equal to* d *with probability one with respect to* $\mathbb{P}^m$*.*

If scenarios are not removed (i.e., $R(S) = \emptyset$) in (2), Campi and Garatti (2008) have shown, under certain technical assumptions, an upper bound on the probability of constraint violation associated to the optimal solution of (2) that is valid for all convex optimisation problems and that holds with equality if the corresponding scenario program is fully-supported. Throughout this paper we consider the following assumption.

Assumption 1 *Problem* (2) *is fully-supported*[3] *and its solution exists and is unique for any* $\{\delta_1, \dots, \delta_m\}$*. Moreover, its feasible set has a non-empty interior.*

Throughout this paper, we consider that the samples in S are ordered, i.e., there exists a bijection $\sigma : \{1, \dots, m\} \to S$, and, for any $i, j \in \{1, \dots, m\}$, $i \neq j$, we say that δ_i is smaller than δ_j whenever $\sigma^{-1}(\delta_i) \leq \sigma^{-1}(\delta_j)$ in the usual sense. Strict inequalities can be used with a similar interpretation. For a fixed $S = \{\delta_1, \dots, \delta_m\}$, we also denote the optimal solution of a scenario program as in (2) for a generic $J \subset S$ as

$$\begin{aligned} z^\star(J) = \underset{x \in \mathcal{X}}{\text{argmin}} \quad & c^\top x \\ \text{subject to} \quad & g(x, \delta) \leq 0, \quad \delta \in J. \end{aligned} \tag{3}$$

2. For a fixed $x \in \mathcal{X}$, the function $g(x, \cdot) : \Delta \to \mathbb{R}$, where $(\Delta, \mathcal{F}, \mathbb{P})$ is a probability measure space, is said to be measurable if for all Borel sets A of $\mathbb{R}$ we have that $g^{-1}(x, A) \in \mathcal{F}$. This implies that the set $\{\delta \in \Delta : g(x, \delta) > 0\} = g^{-1}(x, A)$, with $A = (0, \infty)$, is an element of $\mathcal{F}$, hence rendering (1) well-defined. A Borel set is an element of the σ-algebra generated by the standard topology of $\mathbb{R}$.

3. Most of the results in this paper can be extended to non-fully-supported but non-degenerate problems (see Campi and Garatti (2008) for the definition) using the same technique as in Calafiore (2010) and Romao et al. (2020a), by ordering the samples and creating an augmented (regularised) optimisation problem.

P_k	Removed till $k \in \{0, \dots, q_1\}$	Optimiser at $(k+1)$-th stage
0	$R_0(S) = \emptyset$	$x_0^\star(S)$
1	$R_1(S) = \mathrm{supp}(x_0^\star(S))$	$x_1^\star(S)$
$\vdots$	$\vdots$	$\vdots$
q_1	$R_{q_1}(S) = R_{q_1-1}(S) \cup \mathrm{supp}(x_{q_1-1}^\star(S))$	$x_{q_1}^\star(S)$
$q_1 + 1$	$R_{q_1}(S) \cup \bar{R}(S)$	$x_{q_1+1}^\star(S)$

Table 1: Description of the quantities at the interim stages for the procedure encoded by (4).

We now introduce the removal procedure analysed in Romao et al. (2020a). Let $r < m$ be the number of discarded constraints and write $r = q_1 d + q_2$ using the division algorithm, where q_1 and q_2 are integers and $q_2 < d$. For $k \in \{0, \dots, q_1\}$, consider the sequence of $q_1 + 1$ scenario programs given by

$$P_k : \underset{x \in \mathcal{X}}{\text{minimise}} \quad c^\top x \qquad \text{subject to} \quad g(x, \delta) \leq 0, \quad \delta \in S \setminus R_k(S), \tag{4}$$

where $R_0(S)$ is the empty set, $R_k(S) = R_{k-1}(S) \cup \mathrm{supp}(x_{k-1}^\star(S))$ for $k \in \{1, \dots, q_1\}$, with $x_k^\star(S)$, $k = 0, \dots, q_1$, representing the optimal solution of (4). If q_2 is not equal to zero, we define similarly a scenario program P_{q_1+1} with $R_{q_1+1}(S) = R_{q_1}(S) \cup \bar{R}(S)$, where $\bar{R}(S)$ is a subset of size q_2 from $\mathrm{supp}(x_{q_1}^\star(S))$ containing the q_2-th smallest scenarios according to ordering defined by σ. As the scenario program P_k depends on the solution of the previous stage through $R_k(S)$, this removal scheme can be interpreted as a cascade of $q_1 + 2$ (or $q_1 + 1$, if q_2 is equal to zero) scenarios programs where at each stage the support set associated to the optimal solution is removed and possibly a subset of the support set in the last stage if $k = q_1$ and $q_2 \neq 0$. Each of these quantities are summarised in Table 1. Let

$$x^\star(S) = \begin{cases} x_{q_1}^\star(S), & \text{if } q_2 = 0; \\ x_{q_1+1}^\star(S), & \text{if } q_2 \neq 0. \end{cases} \tag{5}$$

Observe that $x^\star(S)$ is the final decision whose probability of constraint violation we are ultimately interested in.

If $q_2 = 0$, i.e., if the removed scenarios form an integer multiple of d ($r = q_1 d$), then the results in Romao et al. (2020a) allow assessing the probability of constraint violation of $x^\star(S)$. The authors in Romao et al. (2020a) also show that the bound on the probability of constraint violation is tight. To this end, they analyse a class of problems that, roughly speaking, requires removed constraints to be violated by the optimal solution of any scenario program whose constraint are enforced using remaining scenarios. Specifically, Romao et al. (2020a) imposes the following assumption.

Assumption 2 *Let $S = \{\delta_1, \dots, \delta_m\}$ be i.i.d. samples from the unknown probability distribution $\mathbb{P}$ and let $C \subset S$ be any subset of S. For any $k \in \{0, \dots, q_1\}$ if $\delta \in \mathrm{supp}(x_k^\star(C))$, then we have that*

$$g(z^\star(J), \delta) > 0, \text{ for all } J \subset C \setminus (\cup_{j=0}^{k-1} \mathrm{supp}(x_j^\star(C)) \cup \{\delta\}).$$

Under Assumption 2, Romao et al. (2020a) establish the following theorem.

Theorem 3 (Romao et al. (2020a)) *Consider Assumptions 1 and 2. Fix $\epsilon \in (0,1)$, set $r = q_1 d$ and let $m > r + d$. Consider also the scenario discarding scheme as encoded by (4), and note that $x^\star(S) = x^\star_{q_1}(S)$, as $q_2 = 0$. We then have that*

$$\mathbb{P}^m\Big\{(\delta_1, \ldots, \delta_m) \in \Delta^m : \mathbb{P}\big\{\delta \in \Delta : g(x^\star(S), \delta) > 0\big\} > \epsilon\Big\} = \sum_{i=0}^{r+d-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}. \quad (6)$$

Note that the number of removed scenarios is restricted to be a multiple of the dimension of the optimisation problem. Besides, Romao et al. (2020a) proves that (6) holds with inequality for any non-degenerate scenario programs (see Romao et al. (2020a) for more details).

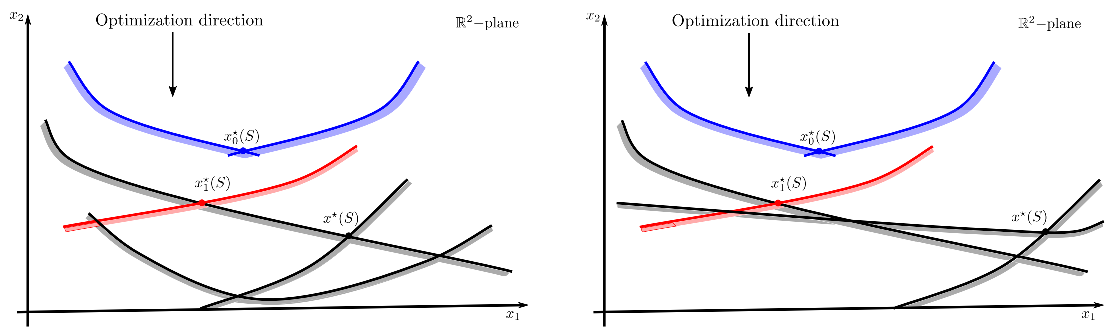
3. Main results

Throughout this section we will explore the removal strategy described by (4) when the number of discarded scenarios is not a multiple of the dimension of the optimisation problem, i.e., when $q_2 \neq 0$. We first show that this is not a straightforward generalisation of the analysis presented in Romao et al. (2020a), as it entails certain difficulties. To this end, consider two realisations of a 2-dimensional ($d = 2$) scenario program as depicted in Figure 1. In both of these realisations our goal is to remove three of the six samples, i.e., we have $q_1 = 1$ and $q_2 = 1$.

We focus first on the realisation shown in Figure 1a. Following the procedure described in (4), the 1st stage removes the scenarios highlighted in blue, as these compose the support set of $x^\star_0(S)$. To remove the third scenario we solve the corresponding scenario program without the scenarios highlighted in blue and obtain $x^\star_1(S)$ as the optimal solution. Assume that the ordering (as detailed in Section 2) is such that the scenario highlighted in red is discarded, thus leading to the solution depicted as $x^\star(S)$. For this realisation, the set composed by the two blue scenarios, the red scenario, and the support set of $x^\star(S)$ constitutes a subset of the samples with cardinality equal to $r + d = 3 + 2 = 5$ such that following the same procedure using only these 5 samples we would obtain the same solutions. Informally, this is related to the notion of compression that will be introduced in the sequel and plays a fundamental role in offering probabilistic feasibility guarantees for the returned solution. Unfortunately, this conclusion is sample dependent and does not hold uniformly across all the samples. For instance, consider the realisation illustrated in Figure 1b. The removal algorithm described in (4) proceeds similarly as in the previous case; however, we notice that the final decision is supported by two scenarios that do not belong to the support set of the previous iteration. This latter fact implies that the cardinality of the subset of the samples that would lead to the same solutions with those that would have been obtained if all the samples were employed is no longer 5 but 6. The difference between these two instances is that in the first one the support sets associated to the two last stages overlap, while in the second one these are disjoint. Moreover, the tighter the bound one can offer the smaller the cardinality of that "representative" subset of the samples. In view of a tight bound, this observation motivates restricting attention to the class of problems where the one of Figure 1a belongs. We formalise this in the next section.

3.1. Arbitrary number of removed scenarios under Assumption 2.

Inspired by the discussion in the previous section, the most natural direction if one wants to produce a tight bound on the resulting decision is that of preventing the situation of Figure 1b to happen. The main result of this section, and also of this paper, is to reveal that this can be obtained by means of the Assumption 2, which was exploited in Romao et al. (2020a) to obtain Theorem 3. To this end, the following proposition is instrumental.



(a) Overlapping support sets. (b) Disjoint support sets.

Figure 1: Two different realisations (1a and 1b) of a two dimensional ($d = 2$) scenario program with six scenarios ($m = 6$) from which three scenarios are discarded ($r = 3$). The scenarios highlighted in blue represent the support set of the 1st stage of the removal procedure, and the ones in red the scenarios removed in the 2nd stage. In realisation 1a the scenario that has not been removed in the 2nd stage belongs to the support set of $x^\star(S)$, which is the optimal solution of the 3rd stage, while in the realisation in 1b none of the remaining scenarios from the 2nd stage belong to the support set of $x^\star(S)$.

Proposition 1 *Consider the removal procedure encoded by (4). Let $S \in \{\delta_1, \dots, \delta_m\}$ be i.i.d. samples from the unknown probability distribution $\mathbb{P}$, and $r = q_1 d + q_2$, with $0 < q_2 < d$. Under Assumptions 1 and 2, if δ is a scenario in $\text{supp}(x^\star_{q_1}(S))$ that has not been removed in the (q_1+1)-th stage, i.e., $\delta \in \text{supp}(x^\star_{q_1}(S)) \setminus \bar{R}(S)$, then δ is in the support set of $\text{supp}(x^\star(S))$.*

Proof Consider the P_{q_1+1} that would arise if $q_2 \neq 0$ and recall that, by (5) , $x^\star(S) = x^\star_{q_1+1}(S)$. Recall also that $R_{q_1+1}(S) = R_{q_1}(S) \cup \bar{R}(S)$, where $\bar{R}(S)$ contains the q_2-th smallest scenarios of $\text{supp}(x^\star_{q_1}(S))$ that will be removed at the $(q_1 + 1)$-th stage. With this in mind, let us prove this proposition by contradiction.

Suppose there exists $\bar{\delta} \in \text{supp}(x^\star_{q_1}(S)) \setminus \bar{R}(S)$ that is not of support for $x^\star(S)$. Such a $\bar{\delta}$ is feasible for problem P_{q_1+1}, i.e., we must have that $g(x^\star(S), \bar{\delta}) \leq 0$. Choose $\bar{J} = \text{supp}(x^\star(S)) \subset S \setminus \{R_{q_1}(S) \cup \{\bar{\delta}\}\}$ (due to the fact that $\bar{\delta} \notin \text{supp}(x^\star(S))$), which then implies that $z^\star(\bar{J}) = x^\star(S)$ and $g(z^\star(\bar{J}), \bar{\delta}) \leq 0$. Under Assumption 2, with $k = q_1$ and since $R_{q_1}(S) = \cup_{j=0}^{q_1-1} \text{supp}(x^\star_j(S))$, the latter is a contradiction, since this would require $g(z^\star(J), \bar{\delta}) > 0$. This concludes the proof of the proposition. ■

In other words, Proposition 1 shows that under Assumption 2 the realisation of Figure 1b can only happen with probability zero. Proposition 1 will be used as the main step to extend the results of Romao et al. (2020a). To achieve this, we follow Romao et al. (2020a) by relying on the concept of compression to establish a probably approximately correct (PAC) bound similar to that of Theorem 3.

Definition 4 (Compression set) *Let $S = \{\delta_1, \dots, \delta_m\}$ be a set of i.i.d. samples from an unknown probability distribution $\mathbb{P}$ and $\mathcal{A} : \Delta^m \to 2^\Delta$ be a mapping. A subset $C \subset S$ with $|C| = \zeta$ is said to be a compression set of cardinality ζ associated to $\mathcal{A}(\cdot)$ if $\delta \in \mathcal{A}(C)$ for all $\delta \in S$.*

In other words, a compression set C contains sufficient information to generate a subset $\mathcal{A}(C)$ that contains all the samples in S. This latter property is called consistency within the learning literature

(Floyd and Warmuth (1995); Vidyasagar (2002)). The notion of compression sets can be used to derive probably approximately correct (PAC) bounds that quantify the confidence with which $\mathcal{A}(C)$ is an approximation of the uncertain space Δ. Not only are we interested in existence but also uniqueness of a compression set related to a mapping $\mathcal{A}(C)$, as these will allow us to characterise an exact bound on the confidence of $\mathcal{A}(C)$ as an approximation for Δ, as summarized below.

Theorem 5 (Theorem 3, **Margellos et al. (2015))** *Fix $S = \{\delta_1, \ldots, \delta_m\}$ and $\epsilon \in (0, 1)$. If there exists a unique compression set of cardinality $\zeta < m$ associated to a mapping $\mathcal{A}(\cdot)$, then*

$$\mathbb{P}^m\{(\delta_1, \ldots, \delta_m) \in \Delta^m : \mathbb{P}\{\delta \in \Delta : \delta \notin \mathcal{A}(C)\} > \epsilon\} = \sum_{i=0}^{\zeta-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}. \tag{7}$$

In the sequel, we identify, under Assumption 2, how the analysis carried in Romao et al. (2020a) can be extended to encompass an arbitrary number of discarded scenarios. We consider the removal strategy described in the previous section. Since all the intermediate problems are fully-supported we remove at each stage the associated support set and in the $(q_1 + 1)$-th stage only a subset of the support set is removed, if q_2 is not zero. Define $\mathcal{A} : \Delta^m \to 2^\Delta$ as

$$\mathcal{A}(C) = \{\delta \in \Delta : g(x^\star(C), \delta) \leq 0\} \cup \left\{ \bigcup_{j=0}^{q_1-1} \mathrm{supp}(x_j^\star(C)) \cup \bigcup_{\delta \in \bar{R}(C)} \delta \right\}, \tag{8}$$

which contains the discarded scenarios in the discrete set, and the set we are ultimately interested in, namely, the set $\{\delta \in \Delta : g(x^\star(C), \delta) \leq 0\}$. Following the algorithmic description presented in (4) a candidate compression set is given by

$$C = \bigcup_{j=0}^{q_1} \mathrm{supp}(x_j^\star(S)) \cup \mathrm{supp}(x^\star(S)), \tag{9}$$

as it contains, under Assumption 1, all the support sets associated to the scenario programs P_k, $k \in \{0, \ldots, q_1 + 1\}$ (due to the fact that $q_2 \neq 0$ and Proposition 1 holds).

Remark 6 *By Proposition 1, we have that* $\mathrm{supp}(x_{q_1}^\star(S)) \cup \mathrm{supp}(x^\star(S)) = \bar{R}(S) \cup \mathrm{supp}(x^\star(S))$, *as any* $\delta \in \mathrm{supp}(x_{q_1}^\star(S))$ *but not in* $\bar{R}(S)$ *will be in* $\mathrm{supp}(x^\star(S))$. *As such,* $|C| = r + d$ *as opposed to* $(q_1 + 2)d$.

Proposition 2 *Consider the removal procedure described by* (4). *Under Assumptions 1 and 2, the set in* (9) *is the unique compression set of size $r + d$ associated to the mapping* (8).

Proof We first prove that (9) is the unique compression for (8) assuming that $x_k^\star(C) = x_k^\star(S)$, for $k \in \{0, \ldots, q_1 + 1\}$.

We start by showing that C is a compression for (8). Let $\bar{\delta}$ be any scenario in S, we need to show that $\bar{\delta} \in \mathcal{A}(C)$. Note that such a $\bar{\delta}$ either belongs to the discrete part of (9), or is feasible to the problem P_{q_1+1}. In the former case, $\bar{\delta}$ is in (9) by definition. In the latter case, it is also in $\mathcal{A}(C)$ since all these scenarios are in $\{\delta : g(x^\star(S), \delta) \leq 0\}$ (since $x^\star(S) = x^\star(C)$). This shows that (9) is a compression set for (8).

Before we proceed to the uniqueness proof, note that $|C| = r + d$ by Remark 6. With this in mind, let C', $C' \neq C$, be another compression of cardinality equal to $r + d$ for the mapping in (8). Let $\bar{k}$ be the minimum k for which $x_k^\star(S) = x_k^\star(C) \neq x_k^\star(C')$. Pick $\bar{\delta} \in \mathrm{supp}(x_{\bar{k}}^\star(S)) \setminus \mathrm{supp}(x_{\bar{k}}^\star(C'))$ such that $\bar{\delta} \in C \setminus C'$, such a $\bar{\delta}$ exists as otherwise we would contradict the fact that $x_{\bar{k}}^\star(S) \neq x_{\bar{k}}^\star(C')$. A similar argument has been used in the proof of Proposition 2, item ii), in Romao et al. (2020a), inspired by Lemma 2.12 of Calafiore (2010). Hence, due to the fact that $\bar{\delta} \notin C \setminus C'$ we have that $\bar{\delta} \notin \mathrm{supp}(x_k^\star(C'))$, for all $k \in \{0, \dots, q_1 + 1\}$, and in particular $\bar{\delta} \notin \mathrm{supp}(x^\star(C'))$. Notice that $\bar{J} = \mathrm{supp}(x^\star(C')) \subset C' \setminus \{\bigcup_{j=0}^{\bar{k}-1} \mathrm{supp}(x_j^\star(S)) \cup \{\bar{\delta}\}\}$ since $x_k^\star(S) = x_k^\star(C')$ for all $k \in \{0, \dots, \bar{k} - 1\}$. As a consequence, by Assumption 2 this would imply that $g(z^\star(\bar{J}), \bar{\delta}) = g(x^\star(C'), \bar{\delta}) > 0$ (recall that $z^\star(\bar{J}) = x^\star(C')$), which contradicts the fact that $\bar{\delta} \in \mathcal{A}(C')$.

To conclude the proof, it remains to be shown that $x_k^\star(S) = x_k^\star(C)$ for any $k \in \{0, \dots, q_1 + 1\}$. This can be done by induction. For $k = 0$, note that $x_0^\star(S) = x_0^\star(C)$ since $\mathrm{supp}(x_0^\star(S)) \subset C$. Suppose $x_k^\star(C) = x_k^\star(S)$ for all $k \in \{0, \dots, \bar{k}\}$, and consider $x_{\bar{k}+1}^\star(S)$. Since $R_{\bar{k}+1}(C) = R_{\bar{k}}(C) \cup \mathrm{supp}(x_{\bar{k}}^\star(S))$ and $R_{\bar{k}}(C) = R_{\bar{k}}(S)$ and $\mathrm{supp}(x_{\bar{k}}^\star(S)) = \mathrm{supp}(x_{\bar{k}}^\star(C))$ by the induction hypothesis, we have that $\mathrm{supp}(x_{\bar{k}+1}^\star(S)) \subset C \setminus R_{\bar{k}+1}(S)$, so $x_{\bar{k}+1}^\star(S) = x_{\bar{k}+1}^\star(C)$. This shows that $x_k^\star(S) = x_k^\star(C)$ for all $k \in \{0, \dots, q_1\}$. In the last stage, where only a subset of the support scenarios is discarded, we can use a similar argument. In fact, as consequence of the fact that $\mathrm{supp}(x_{q_1}^\star(S)) = \mathrm{supp}(x_{q_1}^\star(C))$ we have that $\bar{R}(S) = \bar{R}(C)$. However, this implies that $R_{q_1+1}(C) = R_{q_1+1}(S)$ and then $x_{q_1+1}^\star(C) = x_{q_1+1}^\star(S)$. This concludes the proof of the proposition. ■

The next theorem follows trivially from Proposition 2 and Theorem 5; notice that the result below is tight, i.e., it holds with equality.

Theorem 7 *Consider the removal scheme encoded by* (4) *and suppose Assumptions 1 and 2 hold. Let $S = \{\delta_1, \dots, \delta_m\}$ be i.i.d. samples from the unknown distribution $\mathbb{P}$, $r < m$ be the number of discarded scenarios, and $\epsilon \in (0, 1)$ be given. Write $r = q_1 d + q_2$ and denote as $x^\star(S)$ as in* (5). *Then we have that*

$$\mathbb{P}^m\{(\delta_1, \dots, \delta_m) \in \Delta^m : \mathbb{P}\{\delta \in \Delta : g(x^\star(S), \delta)\} > \epsilon\} = \sum_{i=0}^{r+d-1} \binom{m}{i} \epsilon^i (1 - \epsilon)^{m-i}. \tag{10}$$

Proof The result for $q_2 = 0$ has been proved in Romao et al. (2020a). If $q_2 \neq 0$, then we have by Proposition 2 that a unique compression set exists with cardinality $r + d$. The right-hand side in (10) follows then *mutatis mutandis* from Romao et al. (2020a) with the only difference that the cardinality of the compression set is different. ■

3.2. General removal scheme without Assumption 2

In the previous section we have extended the analysis of the removal algorithm proposed in Romao et al. (2020a) to a general number of discarded scenarios by relying on Assumption 2. In this section, we indicate possible extensions of such procedure without any further restriction on the underlying scenario program, i.e., by relaxing Assumption 2.

The results in Section 3.1 rely on the fact that the cardinality of the set (9) is equal to $r+d$ (which is an immediate consequence of Proposition 1). In fact, without requiring that all the remaining scenarios from the (q_1+1)-th stage are in the support set of the final solution $x^\star(S)$ the tight bound claimed in Theorem 3 does not hold, and the realisation depicted in Figure 1b provides one such instance. We can, however, establish a bound on the probability of constraint violation, but it will no longer be a tight one. This is described in the sequel.

Theorem 8 *Consider the removal scheme described in (4). Let $S = (\delta_1, \ldots, \delta_m)$ be i.i.d. samples from an unknown distribution $\mathbb{P}$ and r be an integer such that $m > \lceil r \rceil_d + d$, where $\lceil r \rceil_d$ is the smallest multiple of d that is larger than r. For any $\epsilon \in (0,1)$ we have that*

$$\mathbb{P}^m\{(\delta_1, \ldots, \delta_m) \in \Delta^m : \mathbb{P}\{\delta \in \Delta : g(x^\star(S), \delta) > 0\} > \epsilon\} \leq \sum_{i=0}^{\lceil r \rceil_d + d - 1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i},$$

Proof See Appendix A. ■

The proof of Theorem 8 follows closely the ones of Theorems 3 and 4 in Romao et al. (2020a) and the proof of Theorem 7 in this paper, i.e., creating a specific mapping that involves the probability of constraint violation and showing that there exists a unique compression set of cardinality equal to $\lceil r \rceil_d + d$ associated to such a mapping. Implicitly, Theorem 8 states that tight results can only be achieved for general scenario programs if scenarios are removed in multiple of the dimension of the optimisation problem.

MIN-MAX SCENARIO PROGRAMS

We can now consider the class of min-max scenario programs. Let $f : \mathcal{X} \times \Delta \to \mathbb{R}$ be a function, where $\mathcal{X}$ and Δ are defined as before. Assume $f(\cdot, \delta)$ is convex for all $\delta \in \Delta$. Given m samples $S = (\delta_1, \ldots, \delta_m)$, we want to solve the following min-max scenario program

$$\min_{x \in \mathcal{X}} \max_{\delta \in S} f(x, \delta),$$

which can be cast, through an epigraphic reformulation, as the following scenario program

$$\begin{aligned} &\underset{(x,t) \in \mathcal{X} \times \mathbb{R}}{\text{minimise}} && t \\ &\text{subject to} && f(x, \delta) \leq t, \text{ for all } \delta \in S. \end{aligned} \tag{11}$$

Consider the following removal scheme, which is inspired by the results in Care et al. (2015) and Garatti et al. (2019) where the empirical cost for min-max scenario programs is characterised. For a given positive integer r such that $m > r + d$ we proceed similarly as in the procedure described by (4), i.e., writing $r = q_1 d + q_2$ and removing $q_1 d$ scenarios by means of a cascade of scenarios programs in which the support set is removed at each stage. However, at the $(q_1 + 1)$-th stage, rather than choosing a subset of size q_2 from $\text{supp}(x^\star_{q_1}(S))$ to be discarded we compute the quantity

$$v_i(S) = t^\star_{q_1}(S) - f(x^\star_{q_1}(S), \delta_i), \text{ for all } \delta_i \in S \setminus \{R_{q_1}(S) \cup \text{supp}(x^\star_{q_1}(S))\}, \tag{12}$$

where $(x^\star_k(S), t^\star_k(S))$, $k \in \{0, \ldots, q_1\}$, is the optimal solution of the scenario program (11), treated as a particular instance of the scenario program (4). It is important to notice that each $v_i(S)$ corresponds to the vertical distance between $t^\star_{q_1}(S)$ and the intersection of the constraint generated by the i-th scenario with the vertical line that passes through $x^\star_{q_1}(S)$ (see Figure 2 for an illustration). We then pick the q_2-th smallest $v_i(S)$ and denote them as $v_{(1)}(S) < v_{(2)}(S) < \ldots < v_{(q_2)}(S)$. The q_2-th layer probability of constraint violation associated to the optimal solution of (11) is then given by (denoting $x^\star_{q_1}(S) = x^\star(S)$)

$$V_{q_2}(S) = \mathbb{P}\{\delta \in \Delta : f(x^\star(S), \delta) > t^\star_{q_1}(S) - v_{(q_2)}(S)\}, \tag{13}$$

which constitutes the probability that an unseen sample has a cost greater than $t^\star_{q_1}(S) - v_{(q_2)}(S)$. An illustration of this procedure for $d = 1, r = 3$, and $m = 9$ is depicted in Figure 2. Under this setting, we can state the following theorem.

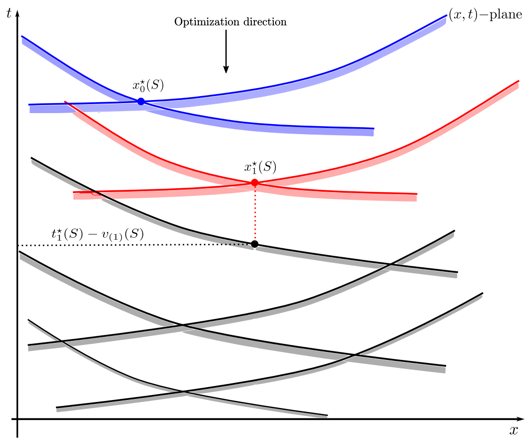


Figure 2: Alternative removal scheme suitable for min-max scenario programs with guaranteed bounds on the probability of constraint violation. We first remove scenarios by removing the support set, and then improve the cost at the last stage by moving downwards, if necessary. The blue and red scenarios correspond to first and second stages of the removal procedure. The dashed-red line defines $v_1(S)$.

Theorem 9 *Consider the removal scheme described in this section and let $V_{q_2}(S)$ be defined as in (13). Let $S = (\delta_1, \dots, \delta_m)$ be i.i.d. samples from an unknown distribution $\mathbb{P}$ and r be an integer such that $m > r + d$. For any $\epsilon \in (0, 1)$ we have that*

$$\mathbb{P}^m\{(\delta_1, \dots, \delta_m) \in \Delta^m : V_{q_2}(S) > \epsilon\} \leq \sum_{i=0}^{r+d-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}.$$

Proof See Appendix B. ■

4. Conclusion

We have analysed how a removal procedure within the sampling-and-discarding approach that yields tight results on the probability of constraint violation of the resulting solution can be extended to an arbitrary number of discarded constraints. Previous results in the literature restricted the number of discarded scenarios to be a multiple of the dimension of the optimisation problem.

In this paper, we have shown that under Assumption 2 the constraint on the number of removed scenarios can be lifted. However, it is elusive how large the class of scenario programs satisfying such an Assumption 2 is. To overcome such a shortcoming we discuss two bounds that allow the decision maker to trade feasibility and performance without requiring further assumptions on the scenario programs. The first one considers the fact that support scenarios may not be shared in the final stage of the procedure, thus leading to a conservative estimate on the resulting probability of violation, which, however, holds uniformly for all samples in Δ^m and for an arbitrary number of discarded scenarios. The second bound holds for the so-called min-max scenario programs and combines our removal strategy with the results in Care et al. (2015) and Garatti et al. (2019).

Acknowledgments

This work was supported by the Coordination for the Improvement of Higher Education Personnel (CAPES), Ministry of Education, Brazil, and by the EPSRC UK under grants EP/P03277X/1 and EP/M002454/1.

Appendix A. Proof of Theorem 8

The proof of Theorem 8 is divided into two steps. We first study the probability of constraint violation associated to the optimal solution of a scenario program for which only a subset of its support scenarios is removed. Then we combine this this analysis with the removal scheme in Romao et al. (2020a) to produce the bound of Theorem 8.

FIRST STEP: REMOVING A SUBSET OF THE SUPPORT SCENARIOS

Recall that we are considering ordered scenarios, and we say that a scenario δ_i is larger than δ_j if $\sigma^{-1}(\delta_i) \leq \sigma^{-1}(\delta_j)$ in the usual sense.

Consider a sequence of two scenario programs as in (2) where one is obtained from the other by removing a subset of the support scenarios. Denote these scenario programs by SC_1 and SC_2, respectively, to distinguish them from the P_k in the removal procedure described in Section 2. Let SC_1 be

$$\begin{aligned} \mathrm{SC}_1 : \underset{x \in \mathcal{X}}{\text{minimise}} \quad & c^\top x \\ \text{subject to} \quad & g(x, \delta) \leq 0, \quad \delta \in S, \end{aligned} \tag{14}$$

where we overlap notation with previous sections of this paper by also denoting S as the input of SC_1. We denote by $v^\star(S)$ the optimal solution of (14) and, as before, by $\mathrm{supp}(v^\star(S))$ its support set. To define SC_2, fix any $0 < q_2 < d$, and let $M(S)$, with $|M(S)| = q_2$, be the subset of $\mathrm{supp}(v^\star(S))$ containing the q_2-th smallest scenarios in $\mathrm{supp}(v^\star(S))$. Then, let SC_2 be

$$\begin{aligned} \mathrm{SC}_2 : \underset{x \in \mathcal{X}}{\text{minimise}} \quad & c^\top x \\ \text{subject to} \quad & g(x, \delta) \leq 0, \quad \delta \in S \setminus M(S), \end{aligned} \tag{15}$$

and denote by $w^\star(S)$ the optimal solution of (15) and $\mathrm{supp}(w^\star(S))$ its support set. To analyse the probability of constraint violation properties associated to $w^\star(S)$ we first define, for an arbitrary set of samples $C \subset S$, the set $N(C)$ containing the $|\mathrm{supp}(v^\star(C)) \cap \mathrm{supp}(w^\star(C))|$-th smallest scenarios from $C \setminus \{\mathrm{supp}(v^\star(C)) \cup \mathrm{supp}(w^\star(C))\}$ and introduce the mapping $\mathcal{B} : \Delta^m \to 2^\Delta$

$$\mathcal{B}(C) = \{\mathcal{B}_1(C) \cap \mathcal{B}_2(C) \cap \mathcal{B}_3(C)\} \cup \bigcup_{\delta_j \in M(C) \cup N(C)} \delta_j, \tag{16}$$

with $\mathcal{B}_1(C) = \{\delta \in \Delta : g(v^\star(C), \delta) \leq 0\}$, $\mathcal{B}_2(C) = \{\delta \in \Delta : g(w^\star(C), \delta) \leq 0\}$,

$$\mathcal{B}_3(C) = \left\{\delta \in \Delta : \delta \geq_\sigma \max_{\xi \in D(C)} \xi\right\} \cup \mathrm{supp}(w^\star(C)).$$

Note that $\mathcal{B}_1(C) \cap \mathcal{B}_2(C)$ contains the scenarios that satisfy both of the interim solutions, namely, $v^\star(C)$ and $w^\star(C)$, while $\mathcal{B}_3(C)$ contains scenarios that are either larger than the maximum scenario[4] in $N(C)$ or the scenarios in $\mathrm{supp}(w^\star(S))$.

4. Formally, the ordering σ^{-1} is only defined on the finite set S, so in principle computing its value for values of δ not in S has void meaning. However, given any finite set S and under mild conditions on the uncertainty space Δ,

Similarly as in the proof of Theorem 7, we establish a guarantee on the probability of constraint violation associated to $w^\star(S)$ by showing that there exists a unique compression set for the mapping in (16). In fact, the next proposition shows that

$$C = \operatorname{supp}(v^\star(S)) \cup \operatorname{supp}(w^\star(S)) \cup \bigcup_{\delta_j \in N(S)} \delta_j \tag{17}$$

is the unique compression of cardinality equal to $2d$ for (16).

Proposition 3 *Let S be a set of $i.i.d.$ scenarios from an unknown probability distribution $\mathbb{P}$ and an integer $0 < q_2 < d$ be given. Consider the cascade of two scenarios programs SC_1 and SC_2 as in (14) and (15), respectively. The following statements hold:*

a) For all m, there is a realisation of scenarios S such that no compression of size smaller than $2d$ exists for the mapping $\mathcal{B}(\cdot)$ in (16).

b) The set C in (17) is the unique compression set of cardinality $2d$ for the mapping $\mathcal{B}(\cdot)$ in (16).

Remark 10 *Proposition 3 states compression properties related to a removal scheme, encoded by SC_1 and SC_2, that discards only a subset of the support scenarios of a scenario program (i.e., the set $M(C)$ above). A striking feature of this scheme is the fact that it may not yield tight bounds on the probability of constraint violation associated to $w^\star(C)$, as we may not have a compression set of cardinality equal to $d + r_2 < 2d$. Such additional conservatism derives from the need to append additional scenarios to compose the set $N(C)$ described above.*

Proof *Item a).* We show the contrapositive statement. Let $S \subset \Delta$ be a set with cardinality m and assume that there exists a compression C' of cardinality $d' < 2d$ for the mapping $\mathcal{B}(\cdot)$ in (16). Fix a realisation S that yields $N(S) = \emptyset$, i.e., one in which the support sets $\operatorname{supp}(v^\star(S))$ and $\operatorname{supp}(w^\star(S))$ are disjoint (e.g., the realisation depicted in Figure (1b)). As the cardinality of C' is strictly smaller than $2d$ we can find a scenario in $\{\operatorname{supp}(v^\star(S)) \cup \operatorname{supp}(w^\star(S))\} \setminus C'$, since the union of the support sets has cardinality equal to $2d$.

Let $\bar\delta$ be an element in $\{\operatorname{supp}(v^\star(S))\cup\operatorname{supp}(w^\star(S))\}\setminus C'$. Such a $\bar\delta$ is either in $\operatorname{supp}(v^\star(S))\setminus C'$ or in $\operatorname{supp}(w^\star(S)) \setminus C'$. Assume that $\bar\delta \in \operatorname{supp}(v^\star(S)) \setminus C'$, then the set $\operatorname{supp}(v^\star(S)) \setminus C'$ is non-empty. We next show that there exists a $\bar\delta \in \operatorname{supp}(v^\star(S)) \setminus C'$ such that $g(v^\star(C'), \bar\delta) > 0$, which then implies that $\operatorname{supp}(v^\star(S))$ must be contained in C'. To this end, suppose for the sake of contradiction that $g(v^\star(C'), \bar\delta) \leq 0$ for all $\bar\delta \in \operatorname{supp}(v^\star(S)) \setminus C'$. This means that $v^\star(C')$ can be obtained by the following scenario program

$$\begin{aligned} \underset{x \in \mathbb{R}^d}{\text{minimise}} \quad & c^\top x \\ \text{subject to} \quad & g(x, \delta) \leq 0, \quad \delta \in C' \cup \operatorname{supp}(v^\star(S)), \end{aligned}$$

as adding the scenarios in $\operatorname{supp}(v^\star(S)) \setminus C'$ does not change the optimal cost. However, by the definition of support set and due to Assumption 1, this implies that $v^\star(C') = v^\star(S)$, which contradicts the fact that $\operatorname{supp}(v^\star(S)) \setminus C'$ is non-empty. Hence, there must be a $\bar\delta \in \operatorname{supp}(v^\star(S)) \setminus C'$ such that

one may extend σ^{-1} to the whole space Δ in a way that its restriction to S is the original bijection, and that renders $\mathcal{B}_3(C)$ measurable.

$g(v^\star(C'), \bar{\delta}) > 0$; however, this contradicts the fact that C' is a compression set for the mapping $\mathcal{B}(\cdot)$ in (16). In other words, if C' is a compression set of cardinality d then $\bar{\delta} \in \mathrm{supp}(w^\star(S)) \setminus C'$.

Since $\mathrm{supp}(v^\star(S)) \subset C'$, we must have that $v^\star(S) = v^\star(C')$ by Assumption 1, which then implies $M(S) = M(C')$. Changing S by $S \setminus \{\mathrm{supp}(v^\star(S)) \cup M(S)\}$ and C' by $C' \setminus \{\mathrm{supp}(v^\star(S)) \cup M(S)\}$ we can argue similarly as above to conclude that if $\mathrm{supp}(w^\star(S)) \setminus C'$ is not empty, then we can find an element in $\bar{\delta} \in \mathrm{supp}(w^\star(S)) \setminus C'$ such that $g(w^\star(C'), \bar{\delta}) > 0$, which contradicts the fact that C' is a compression. This concludes the proof of item a).

Item b). *(Existence)* We start the proof by showing that the set (17) is a compression for the mapping $\mathcal{B}(\cdot)$ in (16). To this end, we need to show that $\delta \in \mathcal{B}(C)$ for all $\delta \in S$. By the choice of C in (17) and under Assumption 1, we note that $v^\star(C) = v^\star(S)$ and $w^\star(C) = w^\star(S)$, which then implies $M(C) = M(S)$ and $N(C) = N(S)$. Pick $\bar{\delta} \in C$ and let us show that $\bar{\delta} \in \mathcal{B}(C)$. Suppose $\bar{\delta} \in \mathrm{supp}(v^\star(S))$. In this case we have two options: (1) either $\bar{\delta} \in M(S)$, which belongs to the discrete part of $\mathcal{B}(C)$; or (2) $\bar{\delta} \notin M(S)$, in which case it can be either in the support of $\mathrm{supp}(w^\star(S))$ or not. If $\bar{\delta} \in \mathrm{supp}(w^\star(S))$, then it belongs to $\mathcal{B}_1(C) \cap \mathcal{B}_2(C) \cap \mathcal{B}_3(C)$. The fact that such a $\bar{\delta}$ belongs to $\mathcal{B}_1(C) \cap \mathcal{B}_2(C)$ is clear due to $g(v^\star(S), \bar{\delta}) \leq 0$ and $g(w^\star(S), \bar{\delta}) \leq 0$, and that $\bar{\delta} \in \mathcal{B}_3(C)$ follows by definition, since $\mathrm{supp}(w^\star(S)) \subset \mathcal{B}_3(C)$. Otherwise, if $\bar{\delta} \in \mathrm{supp}(v^\star(S)) \setminus \mathrm{supp}(w^\star(S))$ then it either belong to $N(S)$, which then belongs to the discrete part of $\mathcal{B}(C)$, or $\bar{\delta} \in \mathrm{supp}(v^\star(S)) \setminus \{\mathrm{supp}(w^\star(S)) \cup N(S)\}$, which then belongs to $\mathcal{B}_1(C) \cap \mathcal{B}_2(C)$ by definition, and to $\mathcal{B}_3(C)$ due to the fact that such a $\bar{\delta}$ must satisfy $\bar{\delta} \geq_\sigma \max_{\xi \in N(S)} \xi$. This shows that $\delta \in \mathcal{B}(C)$ for all $\delta \in \mathrm{supp}(v^\star(C))$.

Suppose now that $\bar{\delta} \in \mathrm{supp}(w^\star(C))$. It is straightforward by means of similar arguments as above to show that $\bar{\delta} \in \mathcal{B}_1(C) \cap \mathcal{B}_2(C) \cap \mathcal{B}_3(C)$, so we have that $\bar{\delta} \in \mathcal{B}(C)$. Last but not least, if $\bar{\delta} \in N(C)$, then it belongs to the discrete part of $\mathcal{B}(C)$. The arguments in the previous two paragraphs show that if $\bar{\delta} \in C$, with C given in (17), then $\bar{\delta} \in \mathcal{B}(C)$.

To conclude the existence part, we need to fix a $\bar{\delta} \in S \setminus C$ and show that $\bar{\delta} \in \mathcal{B}(C)$. Since such a $\bar{\delta}$ is not in the discrete part of the mapping $\mathcal{B}(C)$, we need to show that $\bar{\delta} \in \mathcal{B}_1(C) \cap \mathcal{B}_2(C) \cap \mathcal{B}_3(C)$. As this $\bar{\delta}$ is feasible for both scenarios programs SC_1 and SC_2 we have that $\bar{\delta} \in \mathcal{B}_1(C) \cap \mathcal{B}_2(C)$. It remains to show that $\bar{\delta} \in \mathcal{B}_3(C)$. To this end, note that since $\bar{\delta} \notin C$ we have immediately that $\bar{\delta} >_\sigma \max_{\xi \in N(S)} \xi$, so it belongs to $\mathcal{B}_3(C)$. This show that C given in (17) is a compression set for the mapping $\mathcal{B}(\cdot)$ in (16), thus concluding the existence part of the proof.

(Uniqueness) We divide the uniqueness proof into two cases: $N(S) = \emptyset$ and $N(S) \neq \emptyset$. In the former case, let C' be another compression set of size $2d$. Fix any $\bar{\delta} \in C \setminus C'$ and note that either $\bar{\delta} \in \mathrm{supp}(v^\star(C))$ or $\bar{\delta} \in \mathrm{supp}(w^\star(C))$. If $\bar{\delta} \in \mathrm{supp}(v^\star(S))$ then a similar argument as in item a) (changing S by C in that argument) shows that there exists a $\bar{\delta} \in C \setminus C'$ such that $g(v^\star(C'), \bar{\delta}) > 0$, which contradicts the fact that C' is a compression. A similar argument also holds if $\bar{\delta} \in \mathrm{supp}(w^\star(C))$.

Let us then consider the case in which $N(S) \neq \emptyset$. We proceed similarly as in the previous case by considering C' another compression of size $2d$. Fix any $\bar{\delta} \in C \setminus C'$ and note that $\bar{\delta}$ cannot belong to $\mathrm{supp}(v^\star(C)) \cup \mathrm{supp}(w^\star(S))$, as this would contradict, as before, the fact that C' is a compression. Hence, such a $\bar{\delta}$ must be an element of $N(C) \setminus C'$. Besides, since $\bar{\delta} \notin C'$ and C' is a compression, we must have that $\bar{\delta}$ is in $\mathcal{B}_1(C') \cap \mathcal{B}_2(C') \cap \mathcal{B}_3(C')$. However, $\bar{\delta} \notin \mathcal{B}_3(C')$ as we have $\bar{\delta} <_\sigma \max_{\xi \in N(C')} \xi$, due to the fact that $C' \subset S$, and $\bar{\delta} \notin \mathrm{supp}(w^\star(C')) \subset C'$, thus contradicting the fact that C' is a compression. This concludes the proof of item b). ■

SECOND STEP: COMBINING THE RESULTS IN PROPOSITION 3 WITH THOSE IN ROMAO ET AL. (2020A)

We are now in a position to prove Theorem 8. Its proof relies on Proposition 3 and Theorem 4 in Romao et al. (2020a). Recall that d is the dimension of the optimisation problem P_k and we are writing $r = q_1 d + q_2$, with $0 < q_2 < d$, where $m > \lceil r \rceil_d + d$. Define the mapping $\bar{\mathcal{A}} : \Delta^m \to 2^\Delta$

$$\bar{\mathcal{A}}(C) = \mathcal{A}(C) \cap \{\mathcal{B}(C \setminus R_{q_1}(C)) \cup R_{q_1}(C)\}, \tag{18}$$

where $\mathcal{A}(\cdot)$ is the mapping[5] given by

$$\mathcal{A}(C) = (\mathcal{A}_1(C) \cap \mathcal{A}_2(C)) \cup \mathcal{A}_3(C), \tag{19}$$

with, $\mathcal{A}_1(C) = \{\delta \in \Delta : g(x^\star_{q_1}(S), \delta) \le 0\}$, $\mathcal{A}_3(C) = \bigcup_{k=0}^{q_1-1} \mathrm{supp}(x^\star_k(C))$,

$$\mathcal{A}_2(C) = \left\{ \bigcap_{k=0}^{q_1-1} \left\{ \delta \in \Delta : c^\top z^\star(J \cup \{\delta\}) \le c^\top x^\star_k(S), \text{ for all } J \subset S \setminus R_k(S), \text{ with } |J| = d-1 \right\} \right\}.$$

The mapping $\mathcal{A}(\cdot)$ is associated to the removal procedure encoded by (4) when $q_2 = 0$ and the details of each of its components can be found in Romao et al. (2020a,b), and $\mathcal{B}(\cdot)$ is the mapping of Proposition 3 with input given by $S \setminus R_{q_1}(S)$, rather than S. Note also that under this choice for the input of $\mathcal{B}(\cdot)$ we have $v^\star(S \setminus R_{q_1}(S)) = x^\star_{q_1}(S)$ and $w^\star(S \setminus R_{q_1}(S)) = x^\star_{q_1+1}(S) = x^\star(S)$ (see (5)). In fact, under this notation, the scenarios programs SC_1 and SC_2 in Proposition 3 correspond to the scenario programs P_{q_1} and P_{q_1+1}, respectively, in the description of Section 2.

Once existence and uniqueness of a compression set associated to the mapping $\bar{\mathcal{A}}(\cdot)$ in (18) is established we invoke Theorem 3 in Margellos et al. (2015) to produce the claimed the bound on the probability of constraint violation in Theorem 8. To this end, let us start showing that the subset of the scenarios given by

$$C = \bigcup_{k=0}^{q_1} \mathrm{supp}(x^\star_k(S)) \cup \mathrm{supp}(x^\star(S)) \cup \bigcup_{j \in N(S)} \delta_j \tag{20}$$

is a compression set for the mapping $\bar{\mathcal{A}}(\cdot)$ in (18) – uniqueness will be shown in the sequel. First, note that such a C can be written as

$$C = C_1 \cup C_2,\ C_1 = \bigcup_{k=0}^{q_1} \mathrm{supp}(x^\star_k(S)),\ C_2 = \ \mathrm{supp}(x^\star_{q_1}(S)) \cup \mathrm{supp}(x^\star(S)) \cup \bigcup_{j \in N(S)} \delta_j. \tag{21}$$

The fact that C in (20) forms a compression set for the mapping $\bar{\mathcal{A}}(\cdot)$ follows trivially since C_1 and $C_2(S)$ are compression sets for the removal procedure encoded by (4) due to Theorem 4 in Romao et al. (2020a) and Proposition 1, i.e., $\delta \in \mathcal{A}(C) \cap \{\mathcal{B}(C \setminus R_{q_1}(C)) \cup R_{q_1}(C)\}$ for all $\delta \in S$. Besides, observe that the cardinality of C is equal to $(q_1+2)d = \lceil q_1 d + q_2 \rceil_d + d = \lceil r \rceil_d + d$ due to definition of set $N(S)$ given in Proposition 3 and to the relation $r = q_1 d + q_2$.

We now show that the set C in (20) is the unique compression set of cardinality equal to $\lceil r \rceil_d + d$ for the mapping in (18). Suppose C' is another compression set of cardinality equal to for $\bar{\mathcal{A}}(\cdot)$. This means that $\delta \in \bar{\mathcal{A}}(C')$ for all $\delta \in S$. However, by the results in Romao et al. (2020a), we must

5. We are also overlapping notation here. The mapping $\mathcal{A}(\cdot)$ in this proof is not related to the mapping $\mathcal{A}(\cdot)$ defined in (8).

have $C_1 \subset C'$; otherwise, we would find another compression of size $(q_1+1)d$ for the mapping $\mathcal{A}(\cdot)$ described in Romao et al. (2020a). We also obtain that $\delta \in \mathcal{B}(C')$ for all $\delta \in S$. Since $C' \setminus R_{q_1}(S) \subset S \setminus R_{q_1}(S)$, by Proposition 1, we must also have that $C_2 \subset C$. However, as the cardinality of $C_1 \cup C_2$ is equal to $\lceil r \rceil_d + d$, this implies that $C' = C$, thus show uniqueness of the compression set C in (20).

It remains to be shown how the existence and uniqueness of a compression set for the mapping $\bar{\mathcal{A}}(\cdot)$ can be used to produce the bound of Theorem 8. To this end, recall that (the dependence on S of the inner sets is omitted to simplify the notation)

$$\bar{\mathcal{A}}(S) = \underbrace{\{(\mathcal{A}_1 \cap \mathcal{A}_2) \cup \mathcal{A}_3\}}_{\mathcal{A}(S)} \cap \underbrace{\{(\mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3) \cup \mathcal{B}_4\}}_{\mathcal{B}(S \setminus R_{q_1-1}(S)) \cup R_{q_1}(S)},$$

where we have defined $\mathcal{B}_4 = R_{q_1} \cup \bigcup_{j \in M \cup N} \delta_j$, which contains all the removed scenarios and potentially additional scenarios that compose the set $N(S)$ described in Proposition 3. After some elementary manipulations, we can prove that

$$\begin{aligned} \bar{\mathcal{A}}(S) &\subset (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3) \cup (\mathcal{A}_3 \cup \mathcal{B}_4) \\ &= (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{B}_2 \cap \mathcal{B}_3) \cup (\mathcal{A}_3 \cup \mathcal{B}_4), \end{aligned} \tag{22}$$

where the second equality holds due to the fact that $x^\star_{q_1}(S) = v^\star(S \setminus R_{q_1}(S))$, which in turn implies that $\mathcal{A}_1(S) = \mathcal{B}_1(S \setminus R_{q_1}(S))$. Our ultimate goal is to bound the probability of $\mathcal{B}_2$. With this in mind, we can use (22) to obtain the relation

$$\begin{aligned} &\mathbb{P}^m\{(\delta_1, \ldots, \delta_m) \in \Delta^m : \mathbb{P}\{\delta \notin \mathcal{B}_2(S \setminus R_{q_1-1}(S))\} > \epsilon\} \\ &\qquad \leq \mathbb{P}^m\{(\delta_1, \ldots, \delta_m) \in \Delta^m : \mathbb{P}\{\delta \notin \bar{\mathcal{A}}(S)\} > \epsilon\}. \end{aligned}$$

However, note that the left-hand side of the above inequality is the probability of constraint violation we are interested in and the right-hand side can be upper bounded by Theorem 3 in Margellos et al. (2015) and the fact that there exists a unique compression set of size $\lceil r \rceil_d + d$ (as shown above), yielding the expression in the right-hand side of the inequality in Theorem 8. This concludes the proof of Theorem 8. ■

Appendix B. Proof of Theorem 9

As for the case of Theorem 8, we divide the proof of Theorem 9 into two steps. The first step consists in studying the removal procedure that improves the cost by moving downwards in the direction of the epigraphic variable for the case in which $0 < r < d$, i.e., $q_1 = 0$ and $q_2 = r$. The second step of the proof combines the resulting guarantee with Theorem 4 in Romao et al. (2020a).

THE CASE $q_1 = 0$

Consider the min-max scenario program

$$\begin{aligned} \underset{(x,t) \in \mathcal{X} \times \mathbb{R}}{\text{minimise}} \quad & t \\ \text{subject to} \quad & f(x, \delta) \leq t, \text{ for all } \delta \in S, \end{aligned} \tag{23}$$

where t is called the epigraphic variable, and let $(x^\star(S), t^\star(S))$ be its optimal solution (unique under Assumption 1) and $\text{supp}((x^\star(S), t^\star(S)))$ be its support set. For any $0 < q_2 < d$, define

$v_i(S)$ and $v_{(i)}$ as described in (12), and recall that $v_{(1)}(S) < v_{(2)}(S) < \ldots < v_{(q_2)}(S) < \ldots$. We are interested in the quantity $V_{q_2}(S)$ defined in (13) with $(t^\star_{q_1}(S), x_{q_1}(S)) = (t^\star(S), x^\star(S))$, as we are considering the case $q_1 = 0$, which constitutes the probability that an unseen sample has a cost greater than $t^\star(S) - v_{(q_2)}(S)$.

To produce bounds on the tail distribution of (13) we consider the mapping $\mathcal{B} : \Delta^m \to 2^\Delta$ as

$$\mathcal{B}(C) = \{\delta \in \Delta : f(x^\star(C), \delta) \leq t^\star(C) - v_{(r)}(C)\} \cup C, \tag{24}$$

which is the union of a discrete set containing the samples in C and the set of δ that generates a constraint that intersects the vertical line passing through $x^\star(S)$ below the value given by $t^\star(C) - v_{(r)}(C)$. Our strategy is to show that the set

$$C = \mathrm{supp}((x^\star(S), t^\star(S))) \cup \left\{ \bigcup_{j=1}^{r-1} \delta_{(j)} \right\}, \tag{25}$$

where $\delta_{(j)}$ denotes the scenarios that lead to the j-th largest $v_j(S)$, i.e., $v_{(j)}(S) = t^\star(S) - f(x^\star(S), \delta_{(j)})$, is the unique compression set of cardinality equal to $d + r$ associated to the mapping $\mathcal{B}(\cdot)$ in (24). This is proved in the next proposition.

Proposition 4 *Given a set of samples $S = (\delta_1, \ldots, \delta_m)$. Consider the removal scheme encoded by (23), and the mapping $\mathcal{B}(\cdot)$ as in (24). Let $r = q_2$, with $0 < q_2 < d$ (i.e., $q_1 = 0$), then we have that the set C in (25) is the unique compression set associated to $\mathcal{B}(\cdot)$.*

Proof *(Existence)* We need to show that $\delta \in \mathcal{B}(C)$ for all $\delta \in S$, with C given in (25). Note that if $\delta \in C$, then $\delta \in \mathcal{B}(C)$, as it belongs to the discrete part of the mapping $\mathcal{B}(\cdot)$ in (24). Then, it suffices to show that $\delta \in \mathcal{B}(C)$ for all $\delta \in S \setminus C$. This is trivial since any scenario in $S \setminus C$ leads to $v_{(r)}(C) \leq v_i(C)$, so $f(x^\star(C), \delta_i) \leq t^\star(C) - v_i(C) \leq t^\star(C) - v_{(r)}(C)$. This shows that C in (9) is a valid compression set and its cardinality is equal to $r + d$.

(Uniqueness) To show uniqueness, assume that there exists another compression set C' with cardinality equal to $r + d$. Pick any $\bar{\delta} \in C \setminus C'$. We want to show that $C \setminus C'$ being non-empty contradicts the fact that C' is a compression for the mapping $\mathcal{B}(\cdot)$ in (24).

Suppose that $\bar{\delta} \in \mathrm{supp}((x^\star(S), t^\star(S))) \setminus C'$. By the Definition 1 and Assumption 1, we must have that $(x^\star(S), t^\star(S)) \neq (x^\star(C'), t^\star(C'))$ with $t^\star(C') < t^\star(S)$. We claim that there exists a $\bar{\delta} \in \mathrm{supp}(x^\star(C'), t^\star(C')) \setminus C'$ with the property that

$$t^\star(C') - f(x^\star(C'), \bar{\delta}) \leq 0. \tag{26}$$

Otherwise, if $t^\star(C') - f(x^\star(C'), \bar{\delta}) \geq 0$ for all $\delta \in \mathrm{supp}((x^\star(S), t^\star(S))) \setminus C'$, then $(x^\star(C'), t^\star(C'))$ would be feasible to the scenario program (23) and, since $t^\star(C') < t^\star(S)$, we would contradict optimality of $t^\star(S)$. For such a $\bar{\delta}$ satisfying (26), we now claim that it cannot be in $\mathcal{B}(C')$. To show this, suppose that $\bar{\delta} \in \mathcal{B}(C')$, , then we obtain

$$t^\star(C') \leq f(x^\star(C'), \bar{\delta}) \leq t^\star(C') - v_{(r)}(C'),$$

where the first inequality holds due to (26) and the second one to the fact that $\bar{\delta} \in \mathcal{B}(C')$. However, since $v_{(r)}(C') > 0$ by construction, we reach the contradiction $t^\star(C') < t^\star(C')$. This shows that if C' is a compression set, then $\mathrm{supp}((x^\star(S), t^\star(S))) \subset C'$.

In other words, if C' is a compression and $\bar{\delta} \in C \setminus C'$, we must have that $\bar{\delta} \in \cup_{i=1}^{r-1} \delta_{(i)} \setminus C'$, which implies $(x^\star(C), t^\star(C)) = (x^\star(C'), t^\star(C')) = (x^\star(S), t^\star(S))$. However, since $C' \subset S$ and

$v_{(r)}(S) = v_{(r)}(C)$, we must have that $v_{(r)}(C') > v_{(r)}(C)$. As before, suppose for the sake of contradiction that such a $\bar{\delta}$ belongs to $\mathcal{B}(C')$, then we must have that

$$v_{(r)}(C') \leq t^\star(S) - f(x^\star(S), \bar{\delta}) = v_{(r)}(C),$$

which is a contradiction since $v_{(r)}(C') > v_{(r)}(C)$. Hence, we conclude that C in (25) is the unique compression set of cardinality equal to $(r + d)$ for the mapping $\mathcal{B}(\cdot)$ in (24). This concludes the proof of the proposition. ■

THE GENERAL CASE: $q_1 \neq 0$

To generalise the results to the case $q_1 \neq 0$, we use similar arguments as in the proof of Theorem 8. Indeed, we define $\bar{\mathcal{A}} : \Delta^m \to 2^\Delta$ as

$$\bar{\mathcal{A}}(C) = \mathcal{A}(C) \cap \{\mathcal{B}(C \setminus R_{q_1}(C)) \cup R_{q_1}(C)\}, \tag{27}$$

where the mapping $\mathcal{A}(\cdot)$ is given in (19) and $\mathcal{B}(\cdot)$ is given[6] in (24). As before, the mapping $\mathcal{A}(\cdot)$ represents the first stage of the removal procedure where the support set associated to the scenario program (11) is discarded at each stage. The mapping $\mathcal{B}(\cdot)$ represents the last stage where we move downwards in the direction of the epigraphic variable.

Recall that, given $S = (\delta_1, \ldots, \delta_m)$, we write $r = q_1(d + 1) + q_2$, with $0 < q_2 < d + 1$, using the division algorithm. A compression candidate for the mapping $\bar{\mathcal{A}}(\cdot)$ is

$$C = \bigcup_{k=0}^{q_1} \mathrm{supp}((x_k^\star(S), t_k^\star(S))) \cup \bigcup_{j=1}^{q_2-1} \delta_{(j)}, \tag{28}$$

which has cardinality equal to $(d+1)(q_1+1)+q_2-1 = r+d$. Note that C in (28) can be written as

$$C = C_1 \cup C_2, \text{ with } C_1 = \bigcup_{k=0}^{q_1} \mathrm{supp}((x_k^\star(S), t_k^\star)), \quad C_2 = \mathrm{supp}((x_{q_1}^\star(S), t_{q_1}^\star)) \cup \bigcup_{j=1}^{q_2-1} \delta_{(j)}.$$

To show that C is the unique compression set associated to the mapping $\bar{\mathcal{A}}(\cdot)$ in (27) we follow *mutatis mutandis* the corresponding arguments in the proof of Theorem 8, i.e., using the fact that the mappings $\mathcal{A}(\cdot)$ and $\mathcal{B}(\cdot)$ possess a unique compression set by Theorem 4 in Romao et al. (2020a) and Proposition 4, respectively.

Finally, to assess the risk of the empirical cost, namely, quantity $V_{q_2}(S)$ defined in (13), we use elementary probability arguments on the mappings that compose $\bar{\mathcal{A}}(\cdot)$ in (27) to produce the upper bound

$$\mathbb{P}^m\{(\delta_1, \ldots, \delta_m) \in \Delta^m : V_{q_2}(S) > \epsilon\} \leq \mathbb{P}^m\{(\delta_1, \ldots, \delta_m) \in \Delta^m : \mathbb{P}\{\delta \in \Delta : \delta \notin \bar{\mathcal{A}}(C)\} > \epsilon\}. \tag{29}$$

However, by exploring that there exists a unique compression set of cardinality equal to $r + d$ associated to $\bar{\mathcal{A}}(C)$, we can invoke Theorem 3 in Margellos et al. (2015) to prove the bound in Theorem 9.

6. Observe the overlapping notation. The mapping $\mathcal{B}(\cdot)$ in this section is not related to the mapping $\mathcal{B}(\cdot)$ in (16)

References

Giuseppe Calafiore. Random convex programs. *SIAM Journal on Optimization*, 20(6):3427–3464, 2010.

Giuseppe Calafiore and Marco Campi. Uncertain convex programs: Randomized solutions and confidence levels. *Mathematical Programming*, 102(1):25–46, 2005.

Giuseppe Calafiore and Marco Campi. The scenario approach to robust control design. *IEEE Transactions on Automatic Control*, 51(5):742—-753, 2006.

Marco Campi and Simone Garatti. The exact feasibility of randomized solutions of uncertain convex programs. *SIAM Journal on Optimization*, 19(3):1211–1230, 2008.

Marco Campi and Simone Garatti. A Sampling-and-Discarding Approach to Chance-Constrained Optimization: Feasibility and Optimality. *Journal of Optimization Theory and Applications*, 148: 257–280, 2011.

Aldo Care, Simone Garatti, and Marco Campi. Scenario Min-max Optimization and the Risk of Empirical Costs. *SIAM Journal on Optimization*, 25(4):2061–2080, 2015.

Peyman Esfahani, Tobias Sutter, and John Lygeros. Performance Bounds for the Scenario Approach and an Extension to a Class of Non-Convex Programs. *IEEE Transactions on Automatic Control*, 60(1):46–58, 2015.

Sally Floyd and Manfred Warmuth. Sample compression, learnability, and the Vapnik-Chervonenkis dimension. *Machine Learning*, 21(21):269–304, 1995.

Simone Garatti, Marco Campi, and Algo Carè. On a class of interval predictor models with universal reliability. *Automatica*, 110, 2019.

Kostas Margellos, Maria Prandini, and John Lygeros. On the connection between compression learning and scenario based single-stage and cascading optimization problems. *IEEE Transactions on Automatic Control*, 60(10):2716–2721, 2015.

Arkadi Nemirovski and Alexander Shapiro. Convex approximations of chance constrained programs. *SIAM Journal on Optimization*, 17(4):959–996, 2006.

Bernardo Pagnoncelli, Shabbir Ahmed, and Alexander Shapiro. Sample average approximation method for chance constrained programming: Theory and applications. *Journal of Optimization Theory and Applications*, 142(2):399–416, 2009.

András Prékopa. Probabilistic Programming. 10(C):267–351, 2003.

Licio Romao, Kostas Margellos, and Antonis Papachristodoulou. On the exact feasibility of convex scenario programs with discarded constraints. *arXiv:2003.13433 [math.OC]*, pages 1–15, 2020a.

Licio Romao, Kostas Margellos, and Antonis Papachristodoulou. Tight generalization guarantees for the sampling and discarding approach to scenario optimization. In *59th Conference on Decision and Control. To appear*, 2020b.

Dietmar A. Salamon. *Measure and Integration*. European Mathematical Society, 2016, Zurich, Switzerland, 2016.

M. Vidyasagar. *Learning and Generalization*. Springer, 2002.